Association of Complex Dynamics and Quasi-synchronization Phenomena of Hyperchaotic Dynamical Systems

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***Abstract*— In this paper our aim is to show the viability of associating the complex dynamic of a hyperchaotic master/slave system to a new quasi-synchronized dynamic achieved under a multiplicative transformation over the linear part of the system via the use of a linear controller. The proposed methodology employs simultaneous triangularization in order to ensure certain structure aspects of the system are preserved.**

To illustrate the results we present several examples of well known modified hyperchaotic systems.

Keywords: Control, Synchronization, Nonlinear systems, Output feedback

I. INTRODUCTION

The problem of stability and synchronization preservation has been recently addressed for the case of hyperbolic, nonlinear systems with chaotic dynamics in (Fernández-Anaya *et al.*, 2007) and (Becker-Bessudo *et al.*, 2008). Results reported in these articles deal with strictly linear modifications intended to preserve the hyperbolicity and stability of the system. Based on these results the goal has been to develop further studies in the field of stability and synchronization for modified dynamical systems. One of the advances we have looked into has been the use of non-positive definite matrices to induce modifications in a hyperbolic dynamical system. The pursuit of this line of thought has involved the use of higher dimensional, hyperchaotic systems which may still exhibit some sort of hyperchaotic or chaotic behavior once the system's hyperbolicity has been nullified, thus allowing us to investigate the effects such modifications have on the system's dynamic as well as linear controllers used to synchronize a master/slave configuration.

Another property we are interested in observing is the preservation of hyperchaotic behavior in modified systems. This was achieved through the use of Wolf's method (Wolf *et al.*, 1985) for numerical calculation of a system's Lyapunov exponents. Comparisons between the number of positive and negative Lyapunov exponents before and after the modification will give us a good indication of the system's hyperchaotic behavior.

The intent of this paper is to show some preliminary results derived from ongoing research following these criteria.

II. MATHEMATICAL PRELIMINARIES

In this section we present the necessary definitions that will allow us to prove the main propositions of this paper. The results stem from the use of matrix products of simultaneously triangularizable square matrices and are focused on the task of preserving the complex dynamic of a n-dimensional dynamical system.

Definition 1: Hyperbolic Non-Hyperbolic Associated Dynamics: By this term we understand the following: given an original hyperbolic dynamic present in a distinct system we can induce another particular Non-Hyperbolic dynamic which we can ascribe to a specific modification. For the case of this article we are *associating* the state and error dynamics of a hyperbolic system to those of a non-hyperbolic one induced via a multiplicative matrix M.

Simultaneous triangularization for square matrices is defined as follows

Definition 2: [*S*ee (Lancaster and Tismenetsky, 1985)] The group of matrices A_1, A_2, $\ldots$, A_n is said to be simultaneously triangularizable if there exists a unitary matrix U , where $UU^\top = U^\top U = I$, such that $A_1 = UT_1U^\top$, $A_2 = UT_2U^\top$, $\ldots$, $A_n = UT_nU^\top$, where T_i, for $i = 1\,, 2\,, \ldots\,, n$, are upper triangular matrices and $U^\top$ is the conjugate transpose of matrix U.

For the following discussion consider $\dot{\mathbf{x}} = f(\mathbf{x})$ to be a hyperbolic dynamical system, where $\mathbf{x} \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous differentiable function of its argument and Δ_{psd} as the set of block upper triangular real positive semi-definite matrices. Let $A = \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x}_0}$ be the Jacobian matrix associated with f evaluated at an equilibrium point $\mathbf{x}_0$.

III. Synchronization through the use of Linear Controllers

Consider the following n-dimentional systems in a master-slave configuration, where the master and slave systems are given by

$$\dot{\mathbf{x}}_m = A\mathbf{x}_m + g(\mathbf{x}_m) \quad (1)$$
$$\dot{\mathbf{x}}_s = A\mathbf{x}_s + g(\mathbf{x}_s) + \mathbf{u}(t) \quad (2)$$

where $A \in \mathbb{R}^{n\times n}$ is a constant matrix $\mathbf{x}_m$ and $\mathbf{x}_s$ are the state vectors of the master and slave systems, respectively. $g : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous differentiable, nonlinear function and $\mathbf{u} \in \mathbb{R}^n$ is the control input.

The problem of synchronization considered in this section is the complete-state exact synchronization. That is, the master system and the slave system are synchronized by designing an appropriate nonlinear state feedback control $\mathbf{u}(t)$ which is attached to the slave system such that

$$\lim_{t\to\infty} \|\mathbf{x}_s(t) - \mathbf{x}_m(t)\| \to 0$$

where $\|\cdot\|$ is the Euclidean norm of a vector.

Considering the error state vector $\mathbf{e} = \mathbf{x}_s - \mathbf{x}_m \in \mathbb{R}^n$, $g(\mathbf{x}_s) - g(\mathbf{x}_m) = L(\mathbf{x}_m, \mathbf{x}_s)$ and an error dynamics equation

$$\dot{\mathbf{e}} = A\mathbf{e} + L(\mathbf{x}_m, \mathbf{x}_s) + \mathbf{u}(t)$$

Based in the active control approach (Bai and Lonngren, 2000), to eliminate the nonlinear part of the error dynamics, and choosing $\mathbf{u}(t) = B\mathbf{v}(t) - L(\mathbf{x}_m, \mathbf{x}_s)$, where B is a constant gain matrix which is selected such that (A, B) be controllable (in our case $B = I$), we obtain

$$\dot{\mathbf{e}} = A\mathbf{e} + \mathbf{v}(t).$$

Notice that the original synchronization problem is equivalent to the problem of stabilizing the zero-input solution of the last system by a suitable choice of the state feedback control.

The state-feedback law is given by $\mathbf{v} = -K\mathbf{e}$.

For the needs of this particular article we shall ask that our suitable feedback matrix K be simultaneously triangularizable to A. This state-feedback law renders the error equation to

$$\dot{\mathbf{e}} = (A - K)\mathbf{e} = U(T_A - T_K)U^\top \mathbf{e}$$

with $(A - K)$ a Hurwitz matrix.

A. Simulations of Synchronized Systems

The hyperchaotic Lü system (Chen *et al.*, 2006) is given by the following set of coupled differential equations

$$\dot{x}_1 = -36x_1 + 36x_2 + x_4$$
$$\dot{x}_2 = 20x_2 - x_1x_3$$
$$\dot{x}_3 = -3x_3 + x_1x_2$$
$$\dot{x}_4 = 1.3x_4 + x_1x_3$$

Given that the origin is an equilibrium point for this system we obtain the following Jacobian and feedback matrices

$$A = \begin{bmatrix} -36 & 36 & 0 & 1 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1.3 \end{bmatrix},$$

$$K = \begin{bmatrix} 0.0139 & 0.0089 & 0 & 0.0004 \\ 0.0089 & 40.04 & 0 & 0.0010 \\ 0 & 0 & 0.1623 & 0 \\ 0.0004 & 0.001 & 0 & 2.940 \end{bmatrix}$$

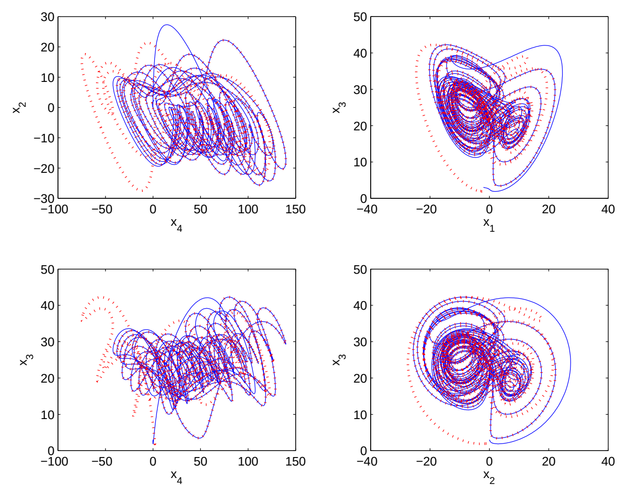

Figure 1. Phase portraits of the synchronized master(—)/slave(···) hyperchaotic Lü system in the $x_4 - x_2$ (top-left), $x_1 - x_3$ (top-right), $x_4 - x_3$ (bottom-left), $x_2 - x_3$ (bottom-right) planes.

The hyperchaotic Rössler system (Arefi and Jahed-Motlagh, 2009) is given by the following set of coupled differential equations

$$\dot{x}_1 = -x_2 + x_4$$
$$\dot{x}_2 = x_1 - 0.25x_2 + x_3$$
$$\dot{x}_3 = -0.05x_3 - 0.5x_4$$
$$\dot{x}_4 = \frac{2}{\sqrt{13}}x_1 + \frac{3\sqrt{13}}{2}x_4 + x_1x_4$$

Given that the origin is an equilibrium point for this system we obtain the following Jacobian and feedback

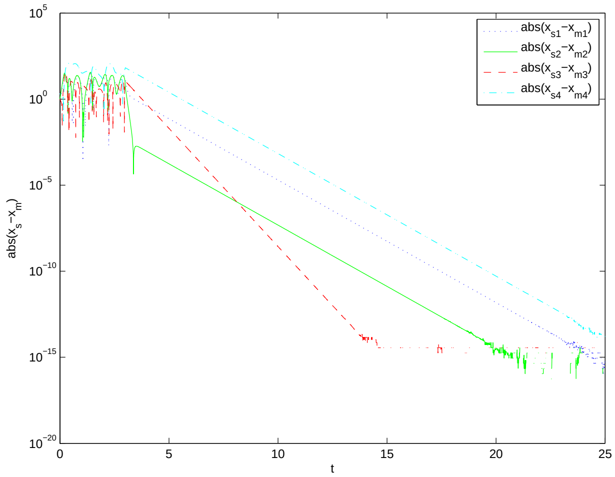

Figure 2. Magnitude of error $|\mathbf{e}| = |\mathbf{x}_s - \mathbf{x}_m|$ between modified master and slave hyperchaotic Lü systems.

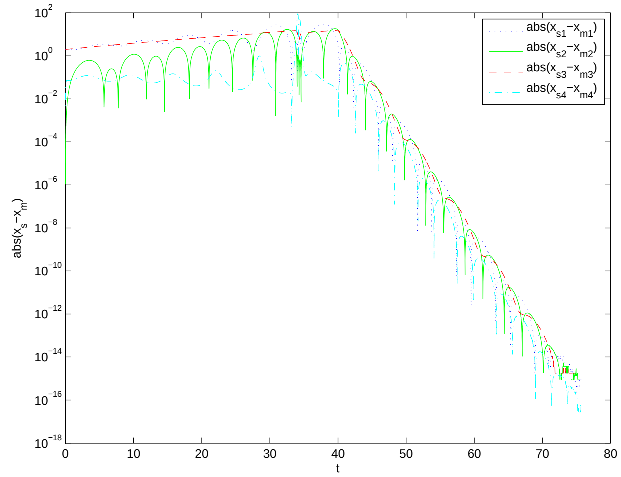

Figure 4. Magnitude of error $|\mathbf{e}| = |\mathbf{x}_s - \mathbf{x}_m|$ between modified master and slave hyperchaotic Rössler systems.

matrices

$$A = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 1/4 & 1 & 0 \\ 0 & 0 & 1/20 & -1/2 \\ 2/\sqrt{13} & 0 & 0 & -3\sqrt{13}/2 \end{bmatrix},$$

$$K = \begin{bmatrix} 0.9437 & 0.0534 & 0.1720 & -0.1465 \\ 0.0534 & 1.123 & 0.4367 & -0.0101 \\ 0.1720 & 0.4367 & 1.331 & -0.1200 \\ -0.1465 & -0.0101 & -0.1200 & 0.1258 \end{bmatrix}$$

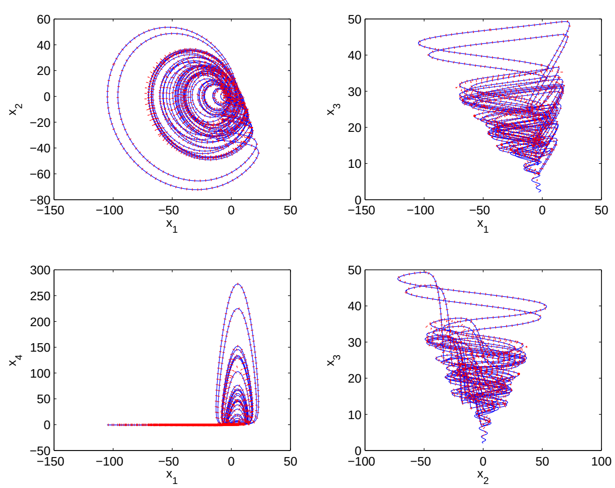

Figure 3. Phase portraits of the synchronized master(—)/slave(···) hyperchaotic Rössler system in the $x_4 - x_2$ (top-left), $x_1 - x_3$ (top-right), $x_4 - x_3$ (bottom-left), $x_2 - x_3$ (bottom-right) planes.

From Figures 1, 2, 3 and 4 we can appreciate the dynamics of both systems and how the applied linear controller synchronizes both master/slave pairs as we confirm the errors' asymptotical trend towards zero.

IV. Quasi-Synchronization in Modified Systems

In this section we will introduce a methodology through which we will induce a modification over the linear part of the nonlinear dynamical system and the error system employing matrices defined within the class Δ_{psd} by means of matrix multiplication.

Definition 3: Constant-Error-Quasi-Synchronization (CEQS) will be understood as the phenomenon where the state errors of the dynamical system remain constant throughout its evolution.

Explicitly this modification will be performed as follows.

Consider a matrix $T_M \in \Delta_{psd}$ and A to be a square matrix, both having the same dimensions. We can decompose A into it's upper triangular form by means of a unitary matrix U, resulting in $A = UT_AU^\top$. Next we will define a new matrix $M = UT_MU^\top$, thus resulting in A and M being simultaneously triangularizable matrices as expressed in definition 2.

After multiplying the linear part of our master/slave system described by (1) and (2) by our constructed modifying M matrix the new system will be defined as follows

$$\dot{\mathbf{x}}_m = MA\mathbf{x}_m + g(\mathbf{x}_m) \quad (3)$$

$$\dot{\mathbf{x}}_s = M(A-K)\mathbf{x}_s + g(\mathbf{x}_s) - L(\mathbf{x}_m, \mathbf{x}_s) \quad (4)$$

by our definition of simultaneously triangularizable matrices and the definition of $L(\mathbf{x}_m, \mathbf{x}_s)$ we can rewrite equations

(3) and (4) as

$$\dot{\mathbf{x}}_m = UT_M T_A U^\top \mathbf{x}_m + g(\mathbf{x}_m) \quad (5)$$
$$\dot{\mathbf{x}}_s = UT_M \left(T_A - T_K\right) U^\top \mathbf{x}_s + g(\mathbf{x}_m) \quad (6)$$

Having done this our new error dynamics equation becomes

$$\dot{\mathbf{e}} = M(A - K)\mathbf{e} = UT_M(T_A - T_K)U^\top \mathbf{e}$$

According to the definition of the set Δ_{psd} we have now transformed our original hyperbolic system into a non-hyperbolic system by changing a certain number of its eigenvalues to exactly 0 by means of the matrix multiplication. This in turn has altered our error dyamics equation now rendering $M(A - K)$ no longer a Hurwitz matrix and thus we can cannot ensure asymptotical stability of the error dynamics. The aim is to see what effect does the transformation have on the original controller and what this does to the errors of the system.

A. Simulations of Associated Quasi-Synchronized Systems

For the Lü system we have defined our modifying matrix M as follows

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rendering our modified Jacobian and feedback matrices as

$$MA = \begin{bmatrix} -36 & 36 & 0 & 1 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$MK = \begin{bmatrix} 0.0139 & 0 & 0 & 0 \\ 0 & 40.04 & 0 & 0 \\ 0 & 0 & 2.940 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In Figure 5 we can appreciate the different sets of trajectories for the modified Lü system, however it is clear that there is no clear evidence of complete state synchronization. However, looking at Figure 6 we see that in fact there are two states which do achieve complete synchronization and two which remain at a constant distance from each other, resulting in CEQS by Definition 3.

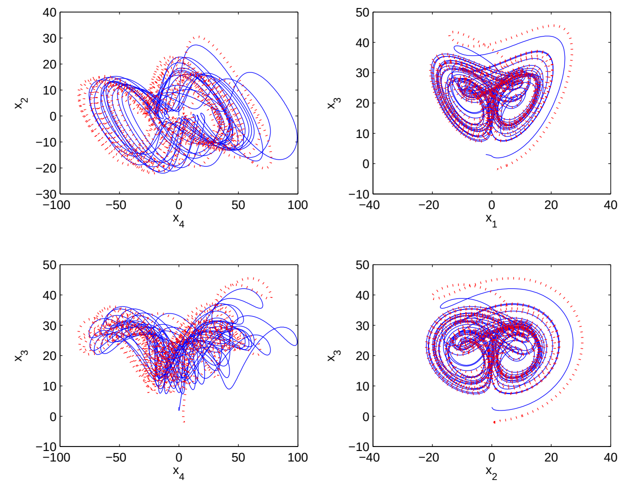

Figure 5. Phase portraits of the modified, quasi-synchronized master(—)/slave(···) Lü system in the $x_4 - x_2$ (top-left), $x_1 - x_3$ (top-right), $x_4 - x_3$ (bottom-left), $x_2 - x_3$ (bottom-right) planes.

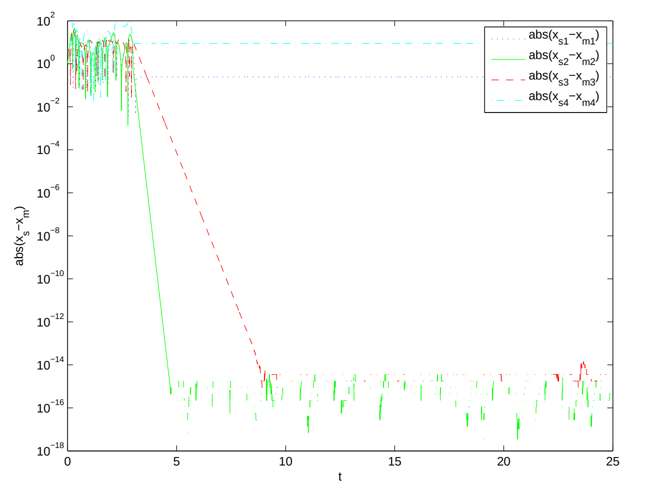

Figure 6. Magnitude of error $|\mathbf{e}| = |\mathbf{x}_s - \mathbf{x}_m|$ between the modified master and slave Lü systems.

For the Rössler system we have defined our modifying matrix M as follows

$$M = \begin{bmatrix} 1.127 & -0.0182 & 2.233 & -0.232 \\ -0.0287 & 1.046 & 0.1803 & -0.0094 \\ 0.0553 & -0.0835 & 0.7371 & 0.0105 \\ 0.6776 & -0.3187 & 8.288 & 0.0897 \end{bmatrix}$$

rendering our modified Jacobian and feedback matrices as

$$MA = \begin{bmatrix} -0.1469 & -1.132 & 0.0934 & -0.9888 \\ 1.041 & 0.2902 & 1.055 & -0.0105 \\ -0.0777 & -0.0761 & -0.0466 & -0.4808 \\ -0.2689 & -0.7573 & 0.0957 & -5.307 \end{bmatrix},$$

$$MK = \begin{bmatrix} 0.9463 & -0.0194 & 3.846 & -0.5127 \\ -0.1173 & 0.9226 & 0.3611 & 0.0280 \\ 0.0506 & -0.0408 & 1.253 & -0.1193 \\ 0.5626 & 0.0755 & 14.12 & -1.322 \end{bmatrix}$$

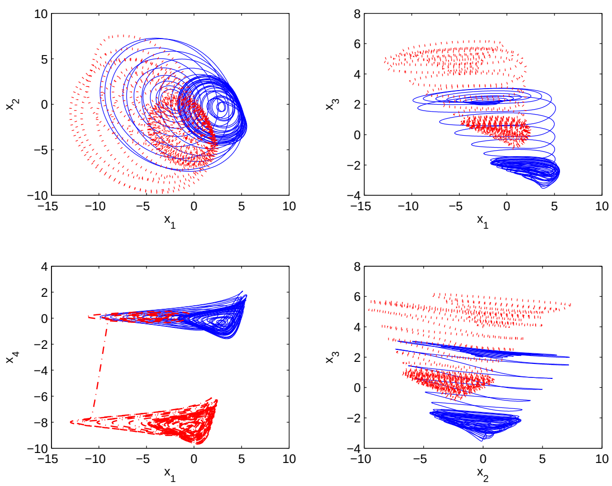

Figure 7. Phase portraits of the modified, desynchronized master(—)/slave($\cdots$) Rössler system in the $x_4 - x_2$ (top-left), $x_1 - x_3$ (top-right), $x_4 - x_3$ (bottom-left), $x_2 - x_3$ (bottom-right) planes.

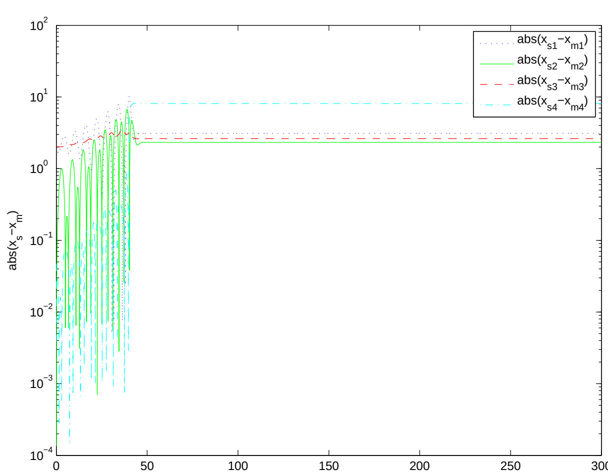

Figure 8. Magnitude of error $|\mathbf{e}| = |\mathbf{x}_s - \mathbf{x}_m|$ between the modified master and slave Rössler systems.

In Figure 7 we can appreciate the different sets of trajectories for the modified Rössler system, however it is clear that there is no evidence of complete state synchronization for any of the master/slave trajectories. Looking at Figure 8 we see that all four states remain at a constant distance from each other after the modified control has been activated, resulting in CEQS across all the states in the system by Definition 3. The modification method described here gives rise to the *association* between the original and modified complex dynamic through the use of the multiplicative matrix M.

V. Lyapunov Exponents

As a way to determine the change in dynamic of the modified system with reference to the original dynamic from which it was constructed we have determined the Lyapunov exponents, using Wolf's method, of both sets of systems shown in Table I

TABLE I

Lyapunov Exponents

	λ_1	λ_2	λ_3	λ_4
Lü	0.9795	0.3181	0	-18.9562
Modified Lü	0.6862	0.0927	0	-19.7081
Rössler	0.1067	0.023	0	-20.8695
Modified Rössler	0.0617	0	-0.1291	-2.7271

From these results we can determine that while for the both Lü systems and the original Rössler system hyperchaos is very likely present for both systems; seeing as they both have two positive Lyapunov exponents, for the modified Rössler system we have a single positive exponent leading us to believe the system is most likely simply chaotic.

VI. Notes on Control Design

As we established in the introduction of this paper, our goal was to identify the *association* between the dynamic of a certain system and that exhibited after a specific modification is carried out over that system.

As the simulations in Figures 6 and 8 show we do not preserve synchronization under these modifications. However, it *is* possible to synchronize the modified systems using the same LQR control design (albeit using the modified system's new structure) as we used for the original system.

Looking at Figures 9, 10, 11 and 12 we see the effect that the new control has over the error dynamics of the modified systems

VII. Conclusions

The concept of association studied in this article is based on the relationship between the proposed modification methodology and the resulting complex dynamic as

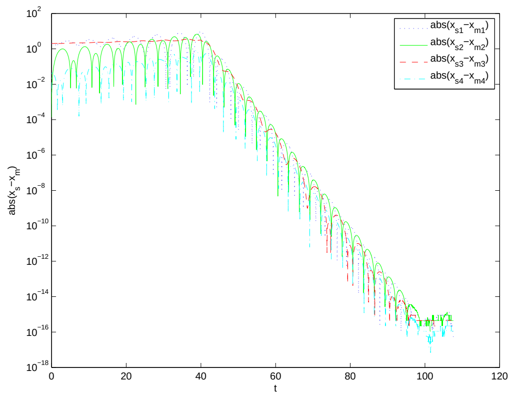

Figure 9. Magnitude of error $|\mathbf{e}| = |\mathbf{x}_s - \mathbf{x}_m|$ between the modified master and slave Rössler systems using the redesigned controller (activated at $t > 40$).

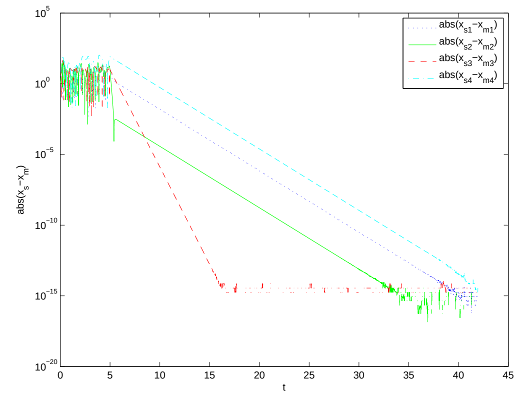

Figure 11. Magnitude of error $|\mathbf{e}| = |\mathbf{x}_s - \mathbf{x}_m|$ between the modified master and slave Lü systems using the redesigned controller (activated at $t > 5$).

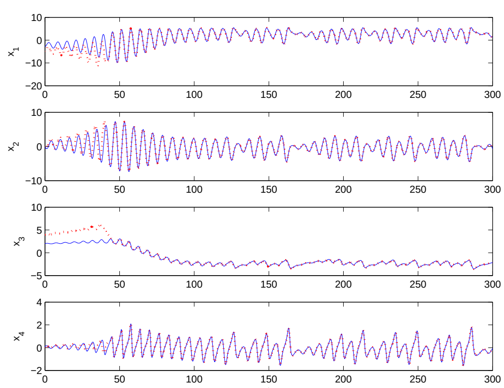

Figure 10. Evolution of each state of the master(—)/slave(···) pair Rössler systems using the redesigned controller (activated at $t > 40$).

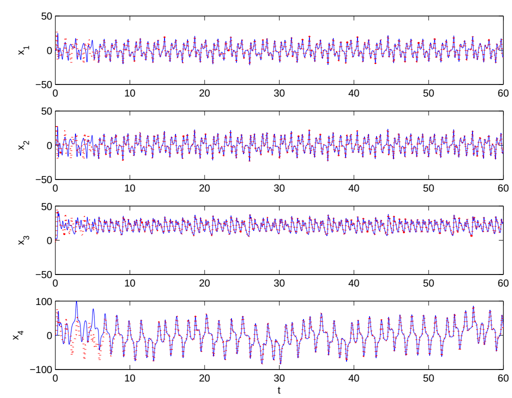

Figure 12. Evolution of each state of the master(—)/slave(···) pair Lü systems using the redesigned controller (activated at $t > 5$).

a consequence of the loss of hyperbolicity. Based on the results found in our research there is evidence to support our claim of the possibility of preserving chaotic/hyperchaotic behavior despite significant changes in the equations that govern a system's dynamic. Despite the results showing that complete state synchronization is not successfully preserved under the modification, we do find different types of quasi-synchronization phenomena as defined in section IV. As seen in section VI if the control is redesigned using the modified system's Jacobian matrix successful synchronization is still achievable. However seeing that this implies the use of a control scheme designed after the modification we cannot speak of these controllers as preserving the original synchronization under the transformation.

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